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Journal of Pure and Applied Algebra 188 (2004) 59–72

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

Stratifying algebras with near-matrix algebras☆

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Received 21 July 2001; received in revised form 27 July 2003

Communicated by E. M. Friedlander

Abstract

Given a left module U and a right module V over an algebra D and a D–D-bilinear form $\beta : U \times V \to D$, we may define an associative algebra structure on the tensor product $V \otimes_D U$. This algebra is called a near-matrix algebra. In this paper, we shall investigate algebras filtered by near-matrix algebras in some nice way and give a unified treatment for quasi-hereditary algebras, cellular algebras, and stratified algebras.

MSC: 16P10; 20G05

1. Introduction

The introduction of quasi-hereditary algebras [2] and stratified algebras [4] gives rise to the following ascending relations between certain classes of finite dimensional algebras:

$$\begin{array}{c}\{\text{semi-simple algebras}\}\\ \cap\\ \{\text{hereditary algebras}\}\\ \cap\\ \{\text{quasi-hereditary algebras}\}\\ \cap\\ \{\text{stratified algebras}\}.\end{array}$$

☆ The authors are supported in part by ARC and NSF.

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doi:10.1016/j.jpaa.2003.10.016

Each class plays some important role in a certain representation theory. For example, hereditary algebras are one of the main objects in the representations theory of quivers and finite dimensional algebras (see e.g., [1]), while the representation theory of quasi-hereditary algebras is closely related to the theory of highest weight categories in Lie theory. Though stratified algebras and their applications are discussed in the monograph [4], it should be interesting to further explore their structure.

In this paper, we attempt to investigate stratified algebras via their "local properties". With the Wedderburn–Artin theorem[1] in mind, we shall use near-matrix algebras defined by tensor products of two modules to filter an algebra in some nice way. This will be called a standard system. The property possessed by these modules are called local properties of the system (or of the algebra). Thus, the subclasses of stratified algebras in the chain:

$$\begin{array}{c}\{\text{standardly stratified algebras}\}\\ \cap\\ \{\text{stratified algebras}\}\\ \cap\\ \{\text{pre-stratified algebras}\}\end{array}$$

roughly correspond to the local properties of projectiveness, flatness and arbitrariness, respectively.

The notion of a standard system of near-matrix algebras is a direct generalization of the notion of a linear standard system.[2] The main obstacle in such a generalization is that the "standard/costandard" modules are no longer free as a module over its endomorphism ring and so the construction cannot rely on using the idea of bases; compare [8,7]. To fix the problem, we use the idea of describing a standard/costandard module as a whole rather than using a basis and view the tensor product of two such modules as an algebra (possibly without identity). These are near-matrix algebras discussed in Section 2. A standard system of an algebra A is now defined as a collection of some near-matrix algebras indexed by a poset and satisfying some axioms. They are discussed in Section 3. We establish the equivalence between bi-free standard systems and linear standard systems in Section 4 and then give in Section 5 an alternative description in terms of filtration of ideals for an algebra with a finite full standard system. These are called pre-stratified algebras. With this result as a preparation, our main results for the characterizations of stratified and standardly stratified algebras are proved in the last section. Finally, as an application of the main results, we prove that for two algebras A and B with stratifications of length n and m, respectively, the tensor product algebra $A \otimes_{\mathbb{k}} B$ has a stratification of length nm, generalizing a result of A. Wiedemann for tensor products of quasi-hereditary algebras.

[1] The theorem asserts that a finite dimensional semi-simple algebra has a decomposition into a direct sum of full matrix algebras defined over some division rings.

[2] We shall call the standard system defined in [5] for characterizing quasi-hereditary algebras a linear standard system.

Throughout, $\mathtt{k}$ is a commutative ring with identity (except for Sections 6 and 7 where $\mathtt{k}$ is a field). By a $\mathtt{k}$-algebra A (or an algebra over $\mathtt{k}$) we mean that A is an associative algebra over $\mathtt{k}$ with identity element 1 unless otherwise specified. We shall denote by ${}_A\mathscr{C}$ (resp. $\mathscr{C}_A$, ${}_A\mathscr{C}_B$) the category of finitely generated left A-modules (resp. right A-modules, A–B-bimodules).

2. Near-matrix algebras

In this section we assume that $\mathtt{k}$ is a commutative ring with 1. Let D be a $\mathtt{k}$-algebra, V a right D-module, and U a left D-module. For a D-bilinear form $\beta : U \times V \to D$, (i.e., β is bilinear and $\beta(xu, v) = x\beta(u, v)$ and $\beta(u, vx) = \beta(u, v)x$ for all $x \in D$, $u \in U$ and $v \in V$,) we define an associative algebra structure (with or without identity) on $V \otimes_D U$ by

$$(v_1 \otimes_D u_1)(v_2 \otimes_D u_2) = v_1\beta(u_1, v_2) \otimes_D u_2 = v_1 \otimes_D \beta(u_1, v_2)u_2.$$

We call this algebra a *near-matrix algebra* and denote it by $(V \otimes_D U, \beta)$. Here, the multiplication of the algebra is defined by β.

The notion of near-matrix algebras is a generalization of the notion of matrix algebras as we shall see from the following.

Lemma 2.1. *Suppose that V and U are, respectively, free right and left D-modules with bases $\{v_1, \ldots, v_n\}$ and $\{u_1, \ldots, u_m\}$. Then the following are equivalent.*

(a) *The near-matrix algebra $M_\beta = (V \otimes_D U, \beta)$ has an identity element*;
(b) *$n = m$ and the matrix $(\beta(u_s, v_t))$ is invertible over D*;
(c) *the near-matrix algebra is isomorphic to the matrix algebra $M_n(D)$.*

Proof. Observe that an element $\sum_{i,j} v_i a_{ij} \otimes_D u_j$ in M_β is an identity element if and only if the matrix (a_{ij}) is the inverse matrix of $(\beta(u_s, v_t))$. In this case, it is necessary that $n = m$. If the matrix $(\beta(u_s, v_t))$ is invertible, then one can choose the bases such that the matrix $(\beta(u_s, v_t))$ is the identity matrix. Then the map $\sum v_i a_{ij} \otimes u_j \mapsto (a_{ij})$ defines an algebra isomorphism from M_β onto the matrix algebra $M_n(D)$ with entries in D. $\square$

Lemma 2.2. *Let $(V \otimes_D U, \beta)$ be a near-matrix algebra and assume that D is Noetherian. Then there exists a near-matrix algebra $(V' \otimes_{D'} U', \beta')$ isomorphic to $(V \otimes_D U, \beta)$ with both V' and U' faithful over D'.*

Proof. Let $J_1 = \mathrm{Ann}_D(V) + \mathrm{Ann}_D(U)$. Set $V_1 = V/VJ_1$, $U_1 = U/J_1U$, and $D_1 = D/J_1$. Then V_1 is a right D/J_1-module and U_1 is a left D/J_1-module. By applying the right exactness of the tensor functors twice and noting that the natural map $VJ_1 \otimes_D U \to V \otimes_D U$ is zero, we get

$$V \otimes_D U \cong V/VJ_1 \otimes_D U \cong V/VJ_1 \otimes_D U/J_1U \cong V_1 \otimes_{D_1} U_1.$$

The above isomorphisms are as abelian groups. The D–D-bilinear map $\beta : U \times V \to D$ induces a bilinear map $\beta_1 : U_1 \times V_1 \to D_1$ naturally by setting $\beta_1(\bar{u}, \bar{v}) = \overline{\beta(u,v)}$ for all $v \in V$ and $u \in U$ with the bar denoting the image in the quotient. Thus, β_1 defines a ring structure on $V_1 \otimes_{D_1} U_1$ and the natural isomorphism $V \otimes_D U \to V_1 \otimes_{D_1} U_1$ of abelian groups actually is multiplicative since

$$(v \otimes_D u)(v' \otimes_D u') = v\beta(u, v') \otimes_D u' \to \bar{v}\beta_1(\bar{u}, \bar{v'}) \otimes \bar{u'} = (\bar{v} \otimes_{D_1} \bar{u})(\bar{v'} \otimes_{D_1} \bar{u'}).$$

Assume that we have constructed U_i and V_i and ideals J_i of D such that $V \otimes_D U \cong V_i \otimes_{D_i} U_i$ as associative algebras for $D_i = D/J_i$, $V_i = V/VJ_i$, $U_i = U/J_iU$, and $\beta_i = \bar{\beta} : U_i \times V_i \to D_i$. We set $J_{i+1} = \mathrm{Ann}_D(V_i) + \mathrm{Ann}_D(U_i)$, $D_{i+1} = D/J_{i+1}$, $V_{i+1} = V/VJ_{i+1}$, $U_{i+1} = U/J_{i+1}U$, and $\beta_{i+1} = \bar{\beta}$. Then we have the isomorphism of algebras $V \otimes_D U \cong V_{i+1} \otimes_{D_{i+1}} U_{i+1}$. We thus obtain a chain of ideals $J_1 \subseteq J_2 \subseteq \cdots$. Set $J = \cup_i J_i$ and $V' = V/VJ$, $U' = U/JU$, and $D' = D/J$. We clearly have $V \otimes_D U \cong V' \otimes_{D'} U'$ as associative algebra by applying the direct limit. We now claim that under the assumption of the lemma, both V' and U' are faithful D'-modules. If D is a Noetherian ring, then there is an integer i_0 such that $J_i = J_{i+1}$ for all $i \geqslant i_0$, i.e., $J_{i+1}/J_i = 0$. Since J_{i+1}/J_i contains $\mathrm{Ann}_{D_i}(V_i) + \mathrm{Ann}_{D_i}(U_i)$, therefore V_i and U_i are both faithful over $D_i = D'$. Note that $V' = V_i$ and $U' = U_i$ for all $i \geqslant i_0$. □

From now on, we always assume that in the definition of the near-matrix algebras, *both V and U are D-faithful.*

Following the definition of the algebra M_β, we have *natural* left and right M_β-module structures on V and U, respectively, defined by

$$(v_1 \otimes_D u_1)v = v_1\beta(u_1, v), \quad (v_1 \otimes_D u_1) = \beta(u, v_1)u_1. \tag{2.2.1}$$

Under the above M_β-module structures, the D-bilinear map $\beta : U \times V \to D$ is M_β-balanced, i.e., $\beta(ua, v) = \beta(u, v')\beta(u', v) = \beta(u, av)$ for all $a = v' \otimes u' \in M_\beta$, $u \in U$ and $v \in V$. Thus β factors through $U \otimes_{M_\beta} V \xrightarrow{\tilde{\beta}} D$ (cf. Remarks 3.4 and 4.3).

Lemma 2.3. *Let $M_\beta = (V \otimes_D U, \beta)$. If β is onto, then the following hold*:

(a) $M_\beta^2 = M_\beta$;
(b) *the natural map $D \to \mathrm{End}_{M_\beta}(V)^{\mathrm{op}}$ is an isomorphism of algebras*;
(c) *the natural map $D \to \mathrm{End}_{M_\beta}(U)$ is an isomorphism of algebras.*

Proof. Since $1 \in \mathrm{Im}(\beta)$, there exist $u_0 \in U$ and $v_0 \in V$ such that $\beta(u_0, v_0) = 1$. Now, for any $u \in U$ and $v \in V$, $v \otimes u = (v \otimes u_0)(v_0 \otimes u) \in M_\beta^2$. Therefore, $M_\beta^2 = M_\beta$, proving (a). Note that β is onto implies that both V and U are D-faithful since $d = \beta(du_0, v_0) = \beta(u_0, v_0 d)$. If $f : V \to V$ is a M_β-module homomorphism, then for any $v \in V$,

$$f(v) = f(v\beta(u_0, v_0)) = f((v \otimes u_0)v_0) = (v \otimes u_0)f(v_0) = v\beta(u_0, f(v_0)).$$

Thus f is in the image of the natural map $D \to \mathrm{End}_{M_\beta}(V)^{\mathrm{op}}$, which is therefore an algebra isomorphism since V is D-faithful. (c) can be proved in a similar way. □

Remark 2.4. One sees easily by modifying the argument slightly that Lemma 2.3 still holds if the condition that β is onto is replaced by the condition that $\tilde{\beta}$ is onto. In fact, if $1=\sum\beta(u_i,v_i)$ for some $u_i\in U$ and $v_i\in V$, then $v\otimes u=\sum(v\otimes u_i)(v_i\otimes u)$ and $f\mapsto\sum\beta(u_i,f(v_i))$ ($f\in\mathrm{End}_{M_\beta}(V)$) and $g\mapsto\sum\beta(g(u_i),v_i)$ ($g\in\mathrm{End}_{M_\beta}(U)$) are respectively the inverse maps of (b) and (c). Conversely, one even can prove that, if both U and V are free on D then $M_\beta^2=M_\beta$ implies that $\tilde{\beta}$ is onto. (For given D-bases, $\{v_1,v_2,\ldots,v_n\}$ and $\{u_1,u_2,\ldots,u_m\}$ for V and U, respectively, write $v_1\otimes u_1$ as a sum of products of elements in M_β. Then equating coefficients proves $1\in\mathrm{Im}(\tilde{\beta})$).

3. Algebras with standard systems

Motivated from [8,7] and [5], we have the following definition.

Definition 3.1. Let A be a $\mathtt{k}$-algebra and Λ a poset. A *standard system* of A is a collection $\mathfrak{C}_\Lambda=\{(V_\lambda\otimes_{D_\lambda}U_\lambda,\beta_\lambda)\}_{\lambda\in\Lambda}$ of near-matrix algebras satisfying the conditions (a), (b) and (c) below.

(a) (Splitting Condition) There are injective $\mathtt{k}$-linear maps (not necessarily algebra homomorphisms)

$$m_\lambda: V_\lambda\otimes_{D_\lambda}U_\lambda\to A$$

such that, $A=\oplus_{\lambda\in\Lambda}A^\lambda$ as $\mathtt{k}$-modules, where $A^\lambda=\mathrm{Im}(m_\lambda)$.

(b) (Order Condition) There are bimodule structures on $V_\lambda\in{}_A\mathscr{C}_{D_\lambda}$ and $U_\lambda\in{}_{D_\lambda}\mathscr{C}_A$ such that

(b1) for any $a\in A$, $v\in V_\lambda$ and $u\in U_\lambda$,

$$am_\lambda(v\otimes u)\equiv m_\lambda(av\otimes u)\bmod(A^{>\lambda}),$$

$$m_\lambda(v\otimes u)a\equiv m_\lambda(v\otimes ua)\bmod(A^{>\lambda}),\qquad\text{where } A^{>\lambda}=\oplus_{\mu>\lambda}A^\mu,$$

(b2) $A^\mu\cdot V_\lambda\neq0$ or $U_\lambda\cdot A^\mu\neq0$ implies $\lambda\geqslant\mu$.

(c) (Associativity Condition) For all $v,v'\in V_\lambda$ and $u,u'\in U_\lambda$,

$$m_\lambda(v'\otimes u')v=v'\beta_\lambda(u',v)\quad\text{and}\quad u'm_\lambda(v\otimes u)=\beta_\lambda(u',v)u.$$

Note that $A^{>\lambda}$ is a two-sided ideal of A. Recall from [5] that a linear standard system consists of linear functions which serve as a “basis” for A and is used to construct “bases” for all standard/costandard modules. In contrast, the shadow of “bases” in the definition of standard systems above disappears. However, we still can view V_λ and U_λ as “standard modules” and (the dual of) “costandard modules”, respectively. On the other hand, we shall see in Section 4 that a linear standard system in the sense of [5] is actually equivalent to the notion of a bi-free standard system below.

Remarks 3.2. (i) We further point out that the Splitting Condition in 3.1 is a modified basis condition, while the first part of the Order Condition corresponds to the condition (C3) in [8, (1.1)]. The Associativity Condition is a compatibility condition between the A-module structure on V_λ and U_λ and the algebra structure on $V_\lambda\otimes_{D_\lambda}U_\lambda$ (cf. (2.2.1)).

This condition and the second part of the Order Condition are automatic for a linear standard system (see [5, (2.6), (3.4)]), and hence for a bi-free standard system, but not in this more general setting.

(ii) If we take $a = m_\lambda(v' \otimes u')$ in (b), then it follows from (c) that

$$\begin{aligned} & m_\lambda(v' \otimes u') \cdot m_\lambda(v \otimes u) \\ & \quad \equiv m_\lambda(m_\lambda(v' \otimes u')v \otimes u) \bmod (A^{>\lambda}) \\ & \quad \equiv m_\lambda(v'\beta_\lambda(u', v) \otimes u) \bmod (A^{>\lambda}) \\ & \quad \equiv m_\lambda((v' \otimes u') \cdot (v \otimes u)) \bmod (A^{>\lambda}). \end{aligned}$$

In particular, we have that, for the natural homomorphism $\pi_\lambda : A \to A/A^{>\lambda}$, the composition map $\pi_\lambda \circ m_\lambda$ is

(a) an algebra monomorphism,
(b) k-split, and
(c) an A–A-bimodule homomorphism.

Furthermore, the D_λ-bilinear map $\beta_\lambda : U_\lambda \otimes V_\lambda \to D_\lambda$ is $m_\lambda(V_\lambda \otimes_{D_\lambda} U_\lambda)$-balanced (though it need not be A-balanced).

It should also be pointed out that the definition of an algebra having a standard system is too general to be of any real use if we don't put any restrictions on the standard system. For example, any $\mathbb{k}$-algebra A is a near-matrix algebra with $V = U = D = A$, and hence, has the obvious trivial (bi-free) standard system in which the bilinear map β is surjective.[3] So, of course, we are interested in standard systems with length ($=|\Lambda|$), say, at least 2. Also, if we impose various conditions on the β_λ's, or on the D_λ's, or on the module structures of V_λ and U_λ, then the algebraic structure of A and its representation theory will change accordingly. For this purpose, we have the following.

Definition 3.3. Let $\mathfrak{C}_\Lambda = \{(V_\lambda \otimes_{D_\lambda} U_\lambda, \beta_\lambda)\}_{\lambda\in\Lambda}$ be a standard system of A.

(a) By a *full* standard system $\mathfrak{C}_\Lambda$, we mean that every β_λ, $\lambda \in \Lambda$, is surjective.
(b) The standard system $\mathfrak{C}_\Lambda$ is called *divisible* (resp., *local*) if every D_λ is a division (resp., local) algebra.
(c) The standard system $\mathfrak{C}_\Lambda$ is called *bi-free* (resp., *bi-projective*), if both V_λ and U_λ are free (resp., projective) over D_λ for all $\lambda \in \Lambda$.
(d) The standard system $\mathfrak{C}_\Lambda$ is called *left projective* (resp., *right projective*) if for every $\lambda \in \Lambda$ the left (resp., right) D_λ-module U_λ (resp. V_λ) is projective.
(e) The standard system $\mathfrak{C}_\Lambda$ is called *flat* if for every $\lambda \in \Lambda$ either the left D_λ-module U_λ or the right D_λ-module V_λ is flat.

[3] This trivial bi-free structure corresponds to the stratifications of topological spaces with a single stratum (the space itself).

(f) The standard system $\mathfrak{C}_\Lambda$ is called Tor-*vanishing* if

$$\mathrm{Tor}_n^{D_\lambda}(V_\lambda, U_\lambda) = 0,$$

for all $n > 0$ and $\lambda \in \Lambda$.

Clearly, a divisible standard system is a bi-free standard system. Also, we have 'descending' relations: (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f).

Remark 3.4. We remark that, unlike the case for a linear standard system [5, (3.3)], the D_λ-bilinear functions $\beta_\lambda : U_\lambda \times V_\lambda \to D_\lambda$ need not be A-balanced in general. So β_λ needs not factor through $U_\lambda \otimes_A V_\lambda \to D_\lambda$ (cf. 4.3(b) below). If β_λ does factor through $U_\lambda \otimes_A V_\lambda \to D_\lambda$, then the onto condition for a full standard system here is stronger than the onto condition for a full linear standard system defined in [5, Section 4]. However, both definitions coincide for local standard systems.

Example 3.5. (1) Every standardly based algebra in the sense of [7] (or an algebra with a linear standard system in the sense of [5]) has a bi-free standard system.

(2) Every algebra with a standard stratification, and hence every quasi-hereditary algebra, has a full and projective standard system. (We give the proofs of these two examples in Sections 4 and 6).

4. Bi-free standard systems

In this section, we first prove the equivalence between the notions of a bi-free standard system and a linear standard system defined in [5].

Recall from [5, 2.1] that, by a linear standard system

$$\mathfrak{c} = \mathfrak{c}(\Lambda;, I, J, D) = \{c_{i,j}^\lambda\}_{\lambda\in\Lambda, i\in I(\lambda), j\in J(\lambda)}$$

of A defined over the data consisting of a poset Λ, index sets $I(\lambda)$ and $J(\lambda)$ and $\mathtt{k}$-algebras $D(\lambda)$ for $\lambda \in \Lambda$, we mean a collection of $\mathtt{k}$-linear injective functions (not necessarily algebra homomorphisms)

$$c_{i,j}^\lambda : D(\lambda) \to A \quad (\lambda\in\Lambda, i\in I(\lambda), j\in J(\lambda)),$$

such that

(1) $A = \oplus_{\lambda\in\Lambda}(\oplus_{i\in I(\lambda), j\in J(\lambda)} c_{i,j}^\lambda(D(\lambda)))$; and
(2) for any $a \in A$ and $x \in D(\lambda)$, we have

$$ac_{i,j}^\lambda(x) \equiv \sum_{i'\in I(\lambda)} c_{i',j}^\lambda(f_{i'}^\lambda(a,i)x) \bmod (A^{>\lambda}),$$

$$c_{i,j}^\lambda(x)a \equiv \sum_{j'\in J(\lambda)} c_{i,j'}^\lambda(xg_{j'}^\lambda(j,a)) \bmod (A^{>\lambda}),$$

where $f_{i'}^{\lambda}(a,i), g_{j'}^{\lambda}(j,a) \in D(\lambda)$ are independent of j and i, respectively, and both are independent of x, and

$$A^{>\lambda} = \bigoplus_{\mu>\lambda} \left(\bigoplus_{i\in I(\mu), j\in J(\mu)} c_{i,j}^{\mu} D(\mu) \right).$$

Theorem 4.1. *An algebra A has a bi-free standard system if and only if A has a linear standard system.*

Proof. If A has a bi-free standard system, then the bimodules $V_\lambda \in {}_A\mathscr{C}_{D_\lambda}$ and $U_\lambda \in {}_{D_\lambda}\mathscr{C}_A$ are D_λ-free. Fix D_λ-bases $\{v_i^\lambda \mid i \in I(\lambda)\}$ and $\{u_j^\lambda \mid j \in J(\lambda)\}$ for V_λ and U_λ, respectively, then we have a $\mathtt{k}$-module decomposition $V_\lambda \otimes_{D_\lambda} U_\lambda = \oplus_{i,j} v_i^\lambda D_\lambda \otimes u_j^\lambda$, where $\otimes = \otimes_{D_\lambda}$ and each direct summand is isomorphic to a copy of D_λ as $\mathtt{k}$-modules. Thus, we have $\mathtt{k}$-linear injective maps from D_λ to $v_i^\lambda D_\lambda \otimes u_j^\lambda$, whose composites with m_λ yield $\mathtt{k}$-linear injective maps:

$$c_{i,j}^{\lambda} : D_\lambda \to A.$$

Thus, Splitting Condition 3.1(a) gives condition (1) above. To check condition (2), we use 3.1(b) and the bimodule structure on V_λ and U_λ to obtain, for $a \in A$ and $x \in D_\lambda$,

$$\begin{aligned} ac_{ij}^{\lambda}(x) &= am_\lambda(v_i x \otimes v_j), \\ &\equiv m_\lambda(a(v_i x) \otimes v_j), \\ &\equiv m_\lambda((av_i)x) \otimes v_j), \\ &\equiv \sum_{i'\in I(\lambda)} m_\lambda(v_{i'} f_{i'}^{\lambda}(a,i)x \otimes u_j), \quad \text{where } av_i = \sum_{i'\in I(\lambda)} v_{i'} f_{i'}^{\lambda}(a,i) \\ &\equiv \sum_{i'\in I(\lambda)} c_{i'j}^{\lambda}(f_{i'}^{\lambda}(a,i)x) \bmod (A^{>\lambda}). \end{aligned}$$

Note that $f_{i'}^{\lambda}(a,i) \in D_\lambda$ is independent of j and x. By a symmetric argument we can also prove the relation for $c_{ij}^{\lambda}(x) \cdot a$. Therefore, $\{c_{ij}^{\lambda}\}$ forms a linear standard system of A.

Conversely, suppose A has a linear standard system. Then ${}_A\mathscr{C}$ (resp. $\mathscr{C}_A$) has standard objects $\Delta(\lambda)$ (resp. $\Delta^{\mathrm{op}}(\lambda)$) as defined in [5, 2.5]. Put $V_\lambda = \Delta(\lambda)$ and $U_\lambda = \Delta^{\mathrm{op}}(\lambda)$. Then the maps m_λ are defined in [5, 3.1] and 3.1(a)–(b) follow immediately from condition (2) above and [5, 2.6]. Note that the Order Condition follows from the facts that $A^\mu V_\lambda \neq 0$ if and only if $A^\mu A^\lambda \not\equiv 0 \,(\mathrm{mod}\, A^{>\lambda})$ and that $A^\mu A^\lambda \subseteq A^{\geqslant\lambda} \cap A^{\geqslant\mu}$ by (2). Now, the bilinear functions β_λ are defined in the first line of the proof of [5, 3.3], and thus, 3.1(c) follows from [5, (3.4)]. Therefore, A has the bi-free standard system $\{(\Delta(\lambda) \otimes_{D(\lambda)} \Delta^{\mathrm{op}}(\lambda), \beta_\lambda)\}_{\lambda\in\Lambda}$. □

Combining this theorem with Remark 3.4 and [5, (5.2)], we have the following.

Corollary 4.2. *Let $\mathtt{k}$ be a field. A finite dimensional k-algebra A is quasi-hereditary if and only if A has a full and divisible standard system.*

Remark 4.3. (a) For a bi-free standard system, the modules V_λ and U_λ are free over D_λ; so, for any fixed $v \in V_\lambda$ and $w \in U_\lambda$ that can be extended to D_λ-bases for V_λ and U_λ, respectively, the left (resp. right) A-module $V_\lambda \otimes w$ (resp. $v \otimes U_\lambda$) is isomorphic to V_λ (resp. U_λ). As in [5], we will denote these modules by $\Delta(\lambda)$ and $\Delta^{\mathrm{op}}(\lambda)$, respectively, and denote the dual $\mathrm{Hom}_{D_\lambda}(\Delta^{\mathrm{op}}(\lambda), D_\lambda)$ by $\nabla(\lambda)$. Note that A-modules V_λ and U_λ actually factor through $A/A^{>\lambda}$. Under the D_λ-freeness condition, the restriction to $A^{\geqslant\lambda}/A^{>\lambda}$ of the $A/A^{>\lambda}$-module structure on V_λ and U_λ is the same as the one defined in (2.2.1).

(b) Following Definition 3.1, for any $v_1, v_2 \in V_\lambda$ and $u_1, u_2 \in U_\lambda$, if $a \in A$, then

$$\begin{aligned}(v_1 \otimes_{D_\lambda} u_1 a)(v_2 \otimes_{D_\lambda} u_2) &\equiv ((v_1 \otimes_{D_\lambda} u_1)a)(v_2 \otimes_{D_\lambda} u_2)\\ &\equiv (v_1 \otimes_{D_\lambda} u_1)(a(v_2 \otimes_{D_\lambda} u_2))\\ &\equiv (v_1 \otimes_{D_\lambda} u_1)(av_2 \otimes_{D_\lambda} u_2) \,(\mathrm{mod}\, A^{>\lambda})\end{aligned}$$

by the associativity. Now using Remark 3.2(ii), we get

$$v_1\beta_\lambda(u_1 a, v_2) \otimes_{D_\lambda} u_2 \equiv v_1\beta_\lambda(u_1, av_2) \otimes_{D_\lambda} u_2 \,(\mathrm{mod}\, A^{>\lambda}), \tag{4.3.1}$$

for all $v_1, v_2 \in V_\lambda$ and $u_1, u_2 \in U_\lambda$. If β_λ is onto, we take u_0 and v_0 such that $\beta_\lambda(u_0, v_0) = 1$. For any u and v, we have

$$v_0 \otimes_{D_\lambda} \beta_\lambda(ua, v)u_0 \equiv v_0 \otimes_{D_\lambda} \beta_\lambda(u, av)u_0 \,(\mathrm{mod}\, A^{>\lambda}). \tag{4.3.2}$$

Let both sides act on u_0 we get $\beta_\lambda(ua, v)u_0 = \beta_\lambda(u, av)u_0$ using the Order Condition. Then $\beta_\lambda(u_1 a, v_2) = \beta_\lambda(u_1, av_2)$ after applying $\beta_\lambda(?, v_0)$. This shows that β_λ is A-balanced and thus factors through $U_\lambda \otimes_A V_\lambda \to D_\lambda$.

5. Pre-stratified algebras

Keep the notations introduced in Section 3. Let A be a $\mathtt{k}$-algebra with a standard system $\mathfrak{C}_\Lambda$. For any $\lambda \in \Lambda$, put $\bar{A}_\lambda = A/A^{>\lambda}$ and $\bar{J}_\lambda = \mathrm{Im}(\pi_\lambda \circ m_\lambda)$, where $\pi_\lambda : A \to \bar{A}_\lambda$ is the natural homomorphism. Clearly, $\bar{J}_\lambda \cong A^\lambda$ as $\mathtt{k}$-modules.

Lemma 5.1. *If β_λ is surjective, then there exists an idempotent $e \in \bar{A}_\lambda$ such that*

(a) $\bar{A}_\lambda e \cong V_\lambda$ *and* $e\bar{A}_\lambda \cong U_\lambda$ *as A-modules*;

(b) $D_\lambda \cong e\bar{A}_\lambda e$; *with the induced D_λ-structures on $\bar{A}_\lambda e$ and $e\bar{A}_\lambda$, the isomorphisms in* (a) *are bimodule isomorphisms*;

(c) *multiplication induces an isomorphism both as A–A-bimodules and as $\mathtt{k}$-algebras*

$$\bar{A}_\lambda e \otimes_{e\bar{A}_\lambda e} e\bar{A}_\lambda \cong \bar{A}_\lambda e \bar{A}_\lambda.$$

Proof. Since β_λ is onto, there are $u_0 \in U_\lambda$ and $v_0 \in V_\lambda$ such that $\beta_\lambda(u_0, v_0) = 1$. Thus, $(v_0 \otimes u_0)^2 = (v_0 \otimes u_0)$ in $V_\lambda \otimes_{D_\lambda} U_\lambda$. Let $e = \pi_\lambda(m_\lambda(v_0 \otimes u_0))$. Then e is an idempotent in $\bar{A}_\lambda$ since $\pi_\lambda \circ m_\lambda$ is multiplicative by 3.2(ii).

For any $v \in V_\lambda$, $v = v\beta_\lambda(u_0, v_0) = m_\lambda(v \otimes u_0)v_0$. Using the Order Condition in 3.1, we have

$$V_\lambda = A^\lambda v_0 = \bar{J}_\lambda v_0 = \bar{A}_\lambda v_0,$$

$$U_\lambda = u_0 A^\lambda = u_0 \bar{J}_\lambda = u_0 \bar{A}_\lambda. \tag{5.1.1}$$

Now, define $f : \bar{A}_\lambda e \to \bar{A}_\lambda v_0$ by sending $\bar{a}e$ to $\bar{a}ev_0$, where $\bar{a} \in \bar{A}_\lambda$. Since $ev_0 = m_\lambda(v_0 \otimes u_0)v_0 = v_0$ by 3.1(c), it follows that f is surjective and $\bar{a}ev_0 = \bar{a}v_0$. Suppose $\bar{a}ev_0 = 0$ with $a \in A$. Then $av_0 = \bar{a}v_0 = 0$ and

$$am_\lambda(v_0 \otimes u_0) \equiv m_\lambda(av_0 \otimes u_0) = 0 \bmod (A^{>\lambda}).$$

So $\bar{a}e = 0$. Therefore, f is an isomorphism of both $\bar{A}_\lambda$-modules and A-modules by the Order Condition. This proves the first isomorphism in (a). The second isomorphism can be proved similarly. Since $\mathrm{End}_A(\bar{A}_\lambda e)^{\mathrm{op}} \cong (e\bar{A}_\lambda e)$, $V_\lambda = \bar{A}_\lambda v_0 \cong \bar{A}_\lambda e$, and

$$D_\lambda \subseteq \mathrm{End}_A(V_\lambda) \subseteq \mathrm{End}_{m_\lambda(V_\lambda \otimes_{D_\lambda} U_\lambda)}(V_\lambda),$$

it follows from 2.3(b) that $\mathrm{End}_A(\bar{A}_\lambda e)^{\mathrm{op}} \cong D_\lambda$, proving (b). Finally, from 2.3(a), we see that $(V_\lambda \otimes_{D_\lambda} U_\lambda)(v_0 \otimes u_0)(V_\lambda \otimes_{D_\lambda} U_\lambda) = (V_\lambda \otimes_{D_\lambda} U_\lambda)$. On the other hand, the isomorphism $\bar{A}_\lambda e \cong V_\lambda$ together with (5.1.1) shows that $\bar{A}_\lambda e = \bar{J}_\lambda e$, and similarly, $e\bar{A}_\lambda = e\bar{J}_\lambda$. Therefore, $\bar{A}_\lambda e \bar{A}_\lambda = \bar{J}_\lambda e \bar{J}_\lambda = \bar{J}_\lambda$ which is isomorphic to $V_\lambda \otimes_{D_\lambda} U_\lambda$. So the statement (c) follows from (a) and (b). □

The following result gives an alternative description of algebras with a finite full standard system.

Theorem 5.2. *Let A be a $\mathtt{k}$-projective algebra. Then A has a finite full standard system if and only if there is a filtration of ideals of A*

$$0 = J_0 \subset J_1 \subset \cdots \subset J_n = A,$$

such that

(a) $\bar{J}_i := J_i/J_{i-1}$ *is* $\mathtt{k}$*-projective*;
(b) *for every i, $\bar{J}_i = \bar{A}_i e_i \bar{A}_i$ for some idempotent $e_i \in \bar{A}_i := A/J_{i-1}$*;
(c) *multiplication induces an isomorphism*

$$\bar{A}_i e_i \otimes_{e_i \bar{A}_i e_i} e_i \bar{A}_i \xrightarrow{\sim} \bar{A}_i e_i \bar{A}_i, \forall i.$$

Proof. Suppose that A has a finite full standard system $\mathfrak{C}_A = \{(V_\lambda \otimes_{D_\lambda} U_\lambda, \beta_\lambda)\}_{\lambda \in \Lambda}$. Then all β_λ are surjective and so 5.1 applies. Choose a linear ordering $\lambda_1, \ldots, \lambda_n$ on Λ ($n = |\Lambda|$) such that $\lambda_i \geqslant \lambda_j \Rightarrow i \leqslant j$ and define $J_i = \oplus_{j \leqslant i} A^{\lambda_j}$. Clearly, by 3.1(b), we have a filtration of ideals

$$0 = J_0 \subset J_1 \subset \cdots \subset J_n = A.$$

Note that 3.1(b) implies that $V_i = V_{\lambda_i}$ (resp., $U_i = U_{\lambda_i}$) is a left (resp., right) module over $\bar{A}_i := \bar{A}_{\lambda_i}$. So, by 5.1 there exists idempotents $e_i \in \bar{A}_i$ such that $J_i/J_{i-1} = \bar{A}_i e_i \bar{A}_i$ and multiplication induces the required isomorphism. Here $J_i/J_{i-1} \cong A^{\lambda_i}$ is $\mathtt{k}$-projective since A^{λ_i} is a $\mathtt{k}$-direct summand of A.

Conversely, for each i, let $V_i=\bar{A}_i e_i$ and $U_i=e_i\bar{A}_i$ and $D_i=e_i\bar{A}_i e_i$ and define $\beta_i: U_i\times V_i\to D_i$ to be the multiplication map. Then β_i is onto ($\beta_i(e_i,e_i)=e_i$), and both V_i and U_i are D_i-faithful. The $\mathtt{k}$-projectivity of $\bar{J}_i=J_i/J_{i-1}$ guarantees that the natural homomorphism $\pi_i: J_i\to\bar{J}_i$ is $\mathtt{k}$-split. Thus, there exists a $\mathtt{k}$-linear map $m_i:\bar{J}_i\to J_i$ such that $J_i=J_{i-1}\oplus m_i(\bar{J}_i)$ and $\pi_i\circ m_i(x)=x$ for all $x\in\bar{J}_i$. Therefore, $A=\oplus_{i=1}^m m_i(\bar{J}_i)$ as $A=\cup_i J_i$. Let $\leqslant'$ be the partial ordering on $\Lambda=\{1,2,\ldots,n\}$ defined as the reversed natural ordering, that is, $i\leqslant' j$ iff $i\geqslant j$. Since $J_i\xrightarrow{\pi_i}\bar{J}_i$ is multiplicative and A-bilinear, then $\pi_i(am_i(x)-m_i(ax))=0$ for all $x\in\bar{J}_i$, proving the relations in 3.1(b1). The relation 3.1(b2) can be seen easily. To show 3.1(c), note that $J_{i-1}\bar{A}_i e_i=0=e_i\bar{A}_i J_{i-1}$. Thus, identifying $xe_i\otimes e_i y$ with $xe_i y$ under the isomorphism in (c),

$$m_i(xe_i\otimes e_i y)ze_i=m_i(xe_i y)ze_i=ze_i yze_i=xe_i\beta_i(e_i y,ze_i)$$

for all $x,y,z\in\bar{A}_i$. So all conditions in 3.1 are satisfied, and hence, the system $\{(V_i\otimes_{D_i}U_i,\beta_i)\}_{1\leqslant i\leqslant n}$ is a (finite) full standard system of A. □

Any algebra A satisfying the condition described in Theorem 5.2 is called a *pre-stratified algebra*. In the next section, we shall see that pre-stratified algebras are crude models for those homologically nicer algebras such as quasi-hereditary or stratified algebras.

6. Projective/flat standard systems and stratified algebras

In this section, we assume that A is a *finite dimensional* algebra over a *field* $\mathtt{k}$. We first recall the definition of stratified algebras introduced in [4].

Definition 6.1. An ideal J of A is called a *stratifying ideal* provided that,

(a) $J=AeA$ for some idempotent $e\in A$;
(b) multiplication induces an isomorphism $Ae\otimes_{eAe}eA\to J$;
(c) $\mathrm{Tor}_n^{eAe}(Ae,eA)=0$ for all $n>0$.

If A has a filtration of ideals

$$0=J_0\subset J_1\subset\cdots\subset J_n=A, \tag{6.1.1}$$

such that J_i/J_{i-1} is a stratifying ideal of A/A_{i-1} for all i, then A is called a *stratified algebra*, and the filtration is called a *stratification* of A (of length n).

By the definition we see that a stratified algebra is a pre-stratified algebra satisfying the homological Tor condition 6.1(c) at every level.

As pointed out in [4, 2.1.2], an ideal J of A is a stratifying ideal if and only if the derived functor $\mathbf{i}_*: D^+({}_{A/J}\mathscr{C})\to D^+({}_A\mathscr{C})$ induced by the exact (inflation) functor $i_*: {}_{A/J}\mathscr{C}\to{}_A\mathscr{C}$ is full embedding, which is equivalent to the following cohomological property:

$$\mathrm{Ext}^\bullet_{{}_{A/J}\mathscr{C}}(M,N)\cong\mathrm{Ext}^\bullet_{{}_A\mathscr{C}}(i_*M,i_*N),\quad \forall M,N\in\mathrm{Ob}({}_{A/J}\mathscr{C}).$$

This definition together with Definition 3.3(f) and Theorem 5.2 yields immediately the following:

Theorem 6.2. *Let A be a finite dimensional algebra over* $\mathtt{k}$. *Then A has a full* Tor-*vanishing standard system $\mathfrak{C}_\Lambda$ if and only if A has a stratification of length $|\Lambda|$. In particular, if A has a flat standard system $\mathfrak{C}_\Lambda$ then A is stratified (with a stratification of length $|\Lambda|$).*

If A has a stratification (6.1.1) and each J_i/J_{i-1} is projective as left (resp., right, both left and right) $\bar{A}_i$-module, then (6.1.1) is called a left standard (resp., right standard, bi-standard) stratification of length n, and A is said to be left standardly (resp., right standardly, bi-standardly) stratified.

Theorem 6.3. *A finite dimensional algebra A has a full left (resp., right) projective standard system $\mathfrak{C}_\Lambda$ if and only if A has a left (resp., right) standard stratification of length $|\Lambda|$.*

Proof. Suppose that A is (left) standardly stratified. Then A has a sequence of ideals

$$0 = J_0 \subset J_1 \subset \cdots \subset J_n = A,$$

such that each J_i/J_{i-1} is idempotent and projective as a left $\bar{A}_i$-module. By the proof given right before [4, (2.1.3)] and Theorem 5.2, we see that A has a full left projective standard system.

Conversely, suppose that A has a full left projective standard system $\mathfrak{C}_\Lambda = \{(V_\lambda \otimes_{D_\lambda} U_\lambda, \beta_\lambda)\}_{\lambda\in\Lambda}$, then, by Theorem 5.2, A has a sequence of ideals

$$0 = J_0 \subset J_1 \subset \cdots \subset J_n = A$$

such that the sections $J_i/J_{i-1} = \bar{A}_i e_i \bar{A}_i$ and $\bar{A}_i e_i \otimes_{e_i\bar{A}_i e_i} e_i\bar{A}_i \cong \bar{A}_i e_i \bar{A}_i$ for some idempotents $e_i \in \bar{A}_i$, where $\bar{A}_i = A/J_{i-1}$. Since U_{λ_i} is a projective D_{λ_i}-module and $U_{\lambda_i} \cong e_i\bar{A}_i$ by Lemma 5.1(a), it follows from Lemma 5.1(b) that $e_i\bar{A}_i$ is a projective $e_i\bar{A}_i e_i$-module. Note that V_{λ_i} is a projective left $\bar{A}_i$-module. The fact that J_i/J_{i-1} is projective as a left $\bar{A}_i$-module follows from the following lemma. □

Lemma 6.4. *If S and T are two rings, M is an S-T-bimodule which is S-projective, and N is a projective left T-module, then $M \otimes_T N$ is a projective left S-module.*

Proof. Using an adjoint pair of functors, we get an isomorphism of functors $\mathrm{Hom}_S(M\otimes_T N, ?) \cong \mathrm{Hom}_T(N, \mathrm{Hom}_S(M, ?))$. The exactness of $\mathrm{Hom}_S(M\otimes_T N, ?)$ follows from the exactness of both $\mathrm{Hom}_T(N, ?)$ and $\mathrm{Hom}_S(M, ?)$. □

We also remark that $M \otimes_T N$ is S-flat if N is T-flat.

Corollary 6.5. *A finite dimensional algebra A has a full bi-projective standard system $\mathfrak{C}_\Lambda$ if and only if A has a bi-standard stratification of length $|\Lambda|$.*

7. Tensor product of standard systems

Assume again in this section that $\mathtt{k}$ is a field, A is a $\mathtt{k}$-algebra with a standard system $\mathfrak{C}_\Lambda = \{(V_\lambda \otimes_{D_\lambda} U_\lambda, \alpha_\lambda)\}_{\lambda\in\Lambda}$, and B is a $\mathtt{k}$-algebra with a standard system $\mathfrak{C}_\Gamma = \{(Y_\gamma \otimes_{E_\gamma} X_\gamma, \beta_\gamma)\}_{\gamma\in\Gamma}$. Note that

$$(V_\lambda \otimes_{D_\lambda} U_\lambda) \otimes_{\mathtt{k}} (Y_\gamma \otimes_{E_\gamma} X_\gamma) \overset{\sigma}{\cong} (V_\lambda \otimes_{\mathtt{k}} Y_\gamma) \otimes_{D_\lambda \otimes_k E_\gamma} (U_\lambda \otimes_{\mathtt{k}} X_\gamma),$$

as $\mathtt{k}$-vector spaces. In fact it is an $A \otimes_{\mathtt{k}} B$-bilinear isomorphism as well. With the $D_\lambda \otimes_{\mathtt{k}} E_\gamma$-bilinear map

$$\alpha_\lambda \otimes \beta_\gamma : (U_\lambda \otimes_{\mathtt{k}} X_\gamma) \times (V_\lambda \otimes_{D_\lambda} Y_\gamma) \to D_\lambda \otimes_{\mathtt{k}} E_\gamma,$$

defined naturally, the above isomorphism σ is also multiplicative. The partial order on $\Lambda\times\Gamma$ is defined by $(\lambda,\gamma) \geqslant (\lambda',\gamma')$ if $\lambda \geqslant \lambda'$ and $\gamma \geqslant \gamma'$. Define $m_{\lambda,\gamma} = m_\lambda \otimes m_\gamma$. Then we have constructed a standard system $\mathfrak{C}_\Lambda \otimes \mathfrak{C}_\Gamma := \{((V_\lambda \otimes_{\mathtt{k}} Y_\gamma) \otimes_{D_\lambda \otimes_{\mathtt{k}} E_\gamma} (U_\lambda \otimes_{\mathtt{k}} X_\gamma), \alpha_\lambda \otimes \beta_\gamma)\}$ for the tensor product algebra $A \otimes_{\mathtt{k}} B$. We call the system $\mathfrak{C}_\Lambda \otimes \mathfrak{C}_\Gamma$ the *tensor product* of $\mathfrak{C}_\Lambda$ and $\mathfrak{C}_\Gamma$.

Let us consider the category $\mathfrak{S}$ with objects being pairs $(A, \mathfrak{C}_\Lambda)$ where A is a $\mathtt{k}$-algebra and $\mathfrak{C}_\Lambda$ is a standard system of A and usual homomorphisms.

Theorem 7.1. *Assume that* $\mathtt{k}$ *is a field. Each of the following full subcategories of* $\mathfrak{S}$ *is closed under tensor product*:

(a) *the subcategory of algebras with full standard systems*;
(b) *the subcategory of algebras with bi-free* (*resp.*, *bi-projective*) *standard systems*;
(c) *the subcategory of algebras with left-projective* (*resp.*, *right-projective*) *standard systems*;
(d) *the subcategory of algebras with flat* (*resp.*, Tor-*vanishing*) *standard systems.*

Proof. (a) is obvious. (b) and (c) follow from the fact if $M \in {}_A\mathscr{C}_D$ and $N \in {}_B\mathscr{C}_E$ are free (or projective) over D and E then $M \otimes_{\mathtt{k}} N \in {}_{A\otimes_{\mathtt{k}} B}\mathscr{C}_{D\otimes E}$ is free (or projective) over $D \otimes E$. To show (d), it is enough to show that for two $\mathtt{k}$-algebras D and E, $V \in \mathscr{C}_D$, $U \in {}_D\mathscr{C}$, $Y \in \mathscr{C}_E$, $X \in {}_E\mathscr{C}$, if $\mathrm{Tor}_n^D(V,U) = 0$ and $\mathrm{Tor}_n^E(Y,X) = 0$ for all $n > 0$, then $\mathrm{Tor}_n^{D\otimes_k E}(V \otimes_{\mathtt{k}} Y, U \otimes_{\mathtt{k}} X) = 0$ for all $n > 0$. In fact, one can take a projective resolution $P^* \to V$ in $\mathscr{C}_D$ and a projective resolution $Q^* \to Y$ in $\mathscr{C}_E$. Then the total complex of the bicomplex $P^* \otimes_{\mathtt{k}} Q^*$ gives a projective resolution of a $V \otimes_{\mathtt{k}} Y$ as $D \otimes_{\mathtt{k}} E$-module. Then apply the functor $- \otimes_{D\otimes_{\mathtt{k}} E} (U \otimes_{\mathtt{k}} X)$ to bicomplex $P^* \otimes_{\mathtt{k}} Q^*$ to obtain the bicomplex $(P^* \otimes_D U) \otimes_{\mathtt{k}} (Q^* \otimes_E X)$ via the natural isomorphism (of $\mathtt{k}$-modules) $(P \otimes_D U) \otimes_{\mathtt{k}} (Q \otimes_E X) \cong (P \otimes_{\mathtt{k}} Q) \otimes_{D\otimes_{\mathtt{k}} E} (U \otimes_{\mathtt{k}} X)$. It now follows from Künneth formula (using the fact that $\mathtt{k}$ is a field) that $\mathrm{Tor}_n^{D\otimes_k E}(V \otimes_{\mathtt{k}} Y, U \otimes_{\mathtt{k}} X) = 0$ for all $n > 0$. $\square$

Corollary 7.2. (1) *For two algebras A and B with stratification of length n and m, respectively, the tensor product algebra* $A \otimes_{\mathtt{k}} B$ *has a stratification of length nm. If both the stratifications of A and B are standard* (*resp. bi-standard*), *so is the tensor* *product. Moreover the standard* (*and co-standard*) *modules for* $A \otimes_{\mathtt{k}} B$ *are tensor product of standard* (*and co-standard*) *modules for* A *and* B.

(2) *If the category* ${}_A\mathscr{C}$ *and* ${}_B\mathscr{C}$ *are stratified by the quasi-posets* Λ *and* Γ, *respectively, then the category* ${}_{A\otimes_{\mathtt{k}}B}\mathscr{C}$ *is stratified by the quasi-poset* $\Lambda \times \Gamma$.

Remark 7.3. (1) Part (c) in the above theorem was proved in [9].

(2) The above theorem works for general commutative ring $\mathtt{k}$ provided that A and B are both $\mathtt{k}$-projective and all modules are $\mathtt{k}$-projective. Thus the theorem in particular applies to orders in semisimple algebras.

(3) The approach of standard systems provides a possible direct way to study integral stratified algebras; compare [3,6].

Acknowledgements

The first author wishes to thank L. Scott for several helpful discussions. The second author thanks the University of New South Wales for its cooperation.

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